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REGULAR PARTIALLY INVARIANT SOLUTIONS OF DEFECT 1 OF THE EQUATIONS OF IDEAL MAGNETOHYDRODYNAMICS

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All irreducible regular partially invariant submodels with one noninvariant function for the equations of ideal magnetohydrodynamics are constructed. The submodels are completed to involution, and partially integrated. The submodels specify Ovsyannikov vortex type motion or motion with homogeneous deformation in some spatial directions.

Key words: *ideal magnetohydrodynamics, partially invariant solutions, overdetermined systems of differential equations.*

Introduction. The notion of a partially invariant solution as a natural generalization of invariant solutions of differential equations was first proposed by Ovsyannikov [1, 2]. The usefulness of this generalization is indicated by numerous examples of partially invariant solutions constructed for the equations of gas dynamics (see [3–6] and [7] and the references therein), hydrodynamics [8–10], the dynamics of a viscous heat-conducting gas [11], magnetohydrodynamics [12, 13], plasticity equations [15, 16], and the equations of other models of mechanics and physics [17–19]. In contrast to invariant submodels, partially invariant submodels are given by overdetermined systems of equations, which complicates their analysis but allows one to obtain classes of solutions with greater arbitrariness compared to invariant solutions.

The general theory of partially invariant solutions of differential equations is set forth in [20]. The notion of the regularity of a partially invariant solution used in practice is introduced in [21]. An important property of partially invariant solutions is reducibility: in some cases, partially invariant solutions coincide with invariant solutions of the same rank. Finding the reduction of a solution is important since this eliminates need to perform the extra work of completing the equations of the submodel to involution. The known sufficient tests of reduction of some special classes of partially invariant solutions are given in [20, 22].

The notion of the hierarchy of partially invariant solutions was introduced in [23] to simplify and systematize the study of the set of partially invariant submodels of a given system of differential equations. The existence of the hierarchical structure reduces the construction of the set of all partially invariant submodels to the analysis of only irreducible submodels, from which all the remaining submodels are obtained by invariant reduction, which considerably simplifies the calculations.

The present paper analyses the regular partially invariant solutions of the equations of ideal magnetohydrodynamics. The analysis is performed only for nonbarochronic submodels in which the pressure is a function of spatial coordinates. The set of barochronic submodels is supposed to be studied separately by analogy with the barochronic submodels in gas dynamics [24]. Eight types of irreducible submodels are identified. The equations of all submodels with one noninvariant function are completed to involution; the obtained equations of the submodel are simpler than the equations of the initial model. The first integrals of the submodels are obtained. It is established that the regular partially invariant submodels for the equations of ideal magnetohydrodynamics are the Ovsyannikov vortex and its generalizations or they correspond to motion with homogeneous deformation over some spatial variables. It should be noted that in the general case, the solutions of the equations magnetohydrodynamics with homogeneous

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deformation over all spatial variables were studied in [25]. In the present work, only a mathematical analysis of the submodels is performed, and the physical content of the solutions is the subject of a separate investigation.

1. Classification of Partially Invariant Solutions. We consider the equations of ideal magnetohydrodynamics [26]

$$D\rho + \rho \operatorname{div} \boldsymbol{u} = 0, \qquad D\boldsymbol{u} + \rho^{-1}\nabla(p + \boldsymbol{B}^2/2) - \rho^{-1}(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} = 0,$$

$$Dp + A(p,\rho) \operatorname{div} \boldsymbol{u} = 0, \qquad D\boldsymbol{B} + \boldsymbol{B} \operatorname{div} \boldsymbol{u} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} = 0, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{B} = 0, \qquad D = \partial_t + \boldsymbol{u}\cdot\nabla.$$

Here $\boldsymbol{u} = (u, v, w)$ is the velocity vector, $\boldsymbol{B} = (H, K, L)$ is the magnetic field vector, and p and ρ are the pressure and density. The equation of state $p = F(S, \rho)$ with entropy S holds. The function $A(p, \rho)$ is given by the equation of state $A = \rho\,(\partial F/\partial\rho)$. All functions depend on time t and Cartesian coordinates $\boldsymbol{x} = (x, y, z)$.

Equations (1.1) admit the 11-dimensional group of transformations G_{11}, which is an extension (by means of homothety) of the Galilean group [18, 27]. The corresponding Lie algebra L_{11} is generated by the following basic operators:

$$X_1 = \partial_x, \qquad X_2 = \partial_y, \qquad X_3 = \partial_z,$$

$$X_4 = t\,\partial_x + \partial_u, \qquad X_5 = t\,\partial_y + \partial_v, \qquad X_6 = t\,\partial_z + \partial_w,$$

$$X_7 = y\,\partial_z - z\,\partial_y + v\,\partial_w - w\,\partial_v + K\,\partial_L - L\,\partial_K,$$

$$X_8 = z\,\partial_x - x\,\partial_z + w\,\partial_u - u\,\partial_w + L\,\partial_H - H\,\partial_L,$$

$$X_9 = x\,\partial_y - y\,\partial_x + u\,\partial_v - v\,\partial_u + H\,\partial_K - K\,\partial_H,$$

$$X_{10} = \partial_t, \qquad X_{11} = t\,\partial_t + x\,\partial_x + y\,\partial_y + z\,\partial_z.$$

By virtue of the structure of the operators of the algebra L_{11}, partially invariant solutions of the equations of magnetohydrodynamics are generated by the same representatives of the optimal system as in the equations of gas dynamics in the absence of a magnetic field. Therefore, these solutions can be constructed using available results [3–6]. Below, we consider only nonbarochronic partially invariant solutions, i.e., solutions in which the pressure p depends on spatial coordinates.

A classification of the irreducible nonbarochronic regular partially invariant solutions of Eqs. (1.1) constructed from subalgebras of the algebra L_{11} is given in [23]. It is shown that the class of irreducible regular nonbarochronic partially invariant submodels for the equations of ideal magnetohydrodynamics (1.1) include the following submodels:
1) submodel $\{X_1, X_4\}$ of rank 3 of defect 1;
2) submodels

$$\{X_2,\ X_3,\ X_7\}, \qquad \{X_5,\ X_6,\ X_7\}, \qquad \{X_7,\ X_8,\ X_9\},$$
$$\{X_3 + X_5,\ X_2 - X_6,\ X_7\}, \qquad \{X_3,\ X_5,\ X_2 + X_6\} \tag{1.2}$$

of rank 2 of defect 1;
3) submodel $\{X_2, X_3, X_5, X_6\}$ of rank 2 of defect 2;
4) submodel $\{X_2, X_3, X_5, X_6, X_7\}$ of rank 2 of defect 3.

All partially invariant submodels of Eqs. (1.1) of defect 1 are constructed below. The equations of each submodel completed to involution and analyzed for irreducibility. In the final form, each submodel is represented by a system of equations with a smaller number of unknowns compared to the initial system. In all cases, the equations for the noninvariant function are completely integrated.

2. Submodel of Rank 3. We consider the partially invariant solution generated by the two-dimensional subalgebra $\{X_1, X_4\}$. The invariants of the subalgebra are all independent variables and unknown functions, except for the first Cartesian coordinate x and the first components of the velocity vector u. To obtain a representation of the solution, we assume that all unknown quantities, except for the function u, depend only on time t and Cartesian coordinates y and z. The corresponding invariant components of the velocity vector will be denoted by V and W. The noninvariant function u will be considered to depend on all initial independent variables: $u = u(t, x, y, z)$.

From the first equation of system (1.1) (the continuity equations) it follows that u depends on x linearly, i.e., it defines motion with homogeneous deformation in the Ox direction (the strain rate tensor does not depend on the coordinate x). For the function u, we choose the following representation:

$$u = xU(t, y, z) + M(t, y, z).$$

Substituting the solution representation into system (1.1) and setting the coefficients of the zero and first powers of the variable x equal to zero, we obtain the following equations of the submodel:

$$\tilde{D}\rho + \rho(U + V_y + W_z) = 0; \tag{2.1a}$$

$$\tilde{D}U + U^2 = 0; \tag{2.1b}$$

$$\tilde{D}M + UM - \rho^{-1}(KH_y + LH_z) = 0; \tag{2.1c}$$

$$\tilde{D}V + \rho^{-1}p_y + \rho^{-1}(HH_y + LL_y - LK_z) = 0; \tag{2.1d}$$

$$\tilde{D}W + \rho^{-1}p_z + \rho^{-1}(HH_z + KK_z - KL_y) = 0; \tag{2.1e}$$

$$\tilde{D}p + A(p, \rho)(U + V_y + W_z) = 0; \tag{2.1f}$$

$$\tilde{D}H + H(V_y + W_z) - KM_y - LM_z = 0; \tag{2.1g}$$

$$\tilde{D}K + K(U + W_z) - LV_z = 0; \tag{2.1h}$$

$$\tilde{D}L + L(U + V_y) - KW_y = 0; \tag{2.1i}$$

$$KU_y + LU_z = 0; \tag{2.1j}$$

$$K_y + L_z = 0. \tag{2.1k}$$

Here $\tilde{D} = \partial_t + V\,\partial_y + W\,\partial_z$. System (2.1) is overdetermined since it contains 11 equations for nine unknown functions. The condition of compatibility of Eqs. (2.1b) and (2.1j) for the function U is the relation

$$U_z(\tilde{D}L - KW_y - LW_z) + U_y(\tilde{D}K - KV_y - LV_z) = 0.$$

In view of (2.1h) and (2.1i), this equation reduces to the following:

$$(KU_y + LU_z)(U + V_y + W_z) = 0.$$

By virtue of (2.1j) the last equation is satisfied identically. Verification of the compatibility of Eq. (2.1k) with the remaining equations of the system shows that the following identity holds:

$$\frac{\partial}{\partial t}\,(2.1\text{k}) = \frac{\partial}{\partial y}\,((2.1\text{h}) - V(2.1\text{k})) + \frac{\partial}{\partial z}\,((2.1\text{i}) - W(2.1\text{k})) - (2.1\text{j}) - U(2.1\text{k}).$$

Here (2.1x) are the left sides of the corresponding equations. Thus, if equality (2.1k) is satisfied for the initial data at $t = 0$, it is also satisfied for the subsequent times. This implies that system (2.1) is an overdetermined system in involution. We show that the overdeterminedness of the system of equations can be simplified by partial integration.

By virtue of (2.1k), for some smooth function ψ, we have

$$K = -\psi_z, \qquad L = \psi_y. \tag{2.2}$$

Substitution of (2.2) into (2.1j) yields the equation $-U_y\psi_z + U_z\psi_y = 0$, which implies the functional dependence $U = U(\psi)$. From Eq. (2.1b), we obtain

$$\Big(\frac{1}{U(\psi)}\Big)'\tilde{D}\psi = 1. \tag{2.3}$$

Here and below, the prime denotes differentiation with respect to ψ. Differentiation of Eq. (2.3) with respect to y and z gives

$$V_y\psi_y + W_y\psi_z + \tilde{D}\psi_y = \Big(\frac{1}{(1/U(\psi))'}\Big)'\psi_y, \qquad V_z\psi_y + W_z\psi_z + \tilde{D}\psi_z = \Big(\frac{1}{(1/U(\psi))'}\Big)'\psi_z.$$

A comparison of the last equations with Eqs. (2.1h) and (2.1i) in the nontrivial case $\psi_y^2 + \psi_z^2 \neq 0$ leads the following equation for the function $U(\psi)$:

$$\Big(\frac{1}{(1/U(\psi))'}\Big)' + U = 0.$$

Integration of this equation yields $U = C_1\,\mathrm{e}^{C_2\psi}$ with arbitrary constants C_1 and C_2. Equation (2.1b) becomes an equation for the function ψ: $\tilde{D}(\mathrm{e}^{-C_2\psi}) = C_1$. Thus, Eqs. (2.1b) and (2.1h)–(2.1k) were integrated. The remaining equations of system (2.1) are a compatible system of equations for all unknown functions.

3. Submodels of Rank 2. There are five submodels (1.2) of rank 2. The submodels $\{X_7, X_8, X_9\}$ and $\{X_2, X_3, X_7\}$ called the Ovsyannikov spherical and plane vortices, are studied in [12] and [13], respectively. The remaining three submodels are considered below.

3.1. *Submodel* $\{X_5, X_6, X_7\}$. This submodel is similar to the Ovsyannikov plane vortex [13]. The representation of the solution is written as

$$u = U(t,x), \quad v = y/t + V(t,x)\cos\omega(t,x,y,z), \quad w = z/t + V(t,x)\sin\omega(t,x,y,z),$$

$$H = H(t,x), \quad K = N\cos(\omega(t,x,y,z) + \sigma(t,x)), \quad L = N\sin(\omega(t,x,y,z) + \sigma(t,x)), \tag{3.1}$$

$$p = p(t,x), \qquad \rho = \rho(t,x).$$

We substitute representation (3.1) into system (1.1). By virtue of the continuity equation, we introduce the auxiliary invariant function $h(t,x)$ defined by the equation

$$\tilde{D}\rho + \rho(U_x + 2/t + hV) = 0, \qquad \tilde{D} = \partial_t + U\,\partial_x. \tag{3.2}$$

From the remaining part of the continuity equation, the noninvariant function ω is expressed as

$$\sin\omega\,\omega_y - \cos\omega\,\omega_z + h = 0. \tag{3.3}$$

From the first components of the vector equations of the momentum and induction and from the equation for pressure, we obtain relations that contain only invariant functions:

$$\tilde{D}U + \rho^{-1}p_x + \rho^{-1}NN_x = 0,$$

$$\tilde{D}H + H(2t^{-1} + hV) = 0, \tag{3.4}$$

$$\tilde{D}p + A(p,\rho)(U_x + 2t^{-1} + hV) = 0.$$

The remaining equations of system (1.1) form an overdetermined system for the function ω. Making nondegenerate linear combinations of the momentum equations in projections onto the Oy and Oz axes, we obtain

$$\rho V\omega_t + (\rho UV - HN\cos\sigma)\omega_x + (\rho V(yt^{-1} + V\cos\omega) - N^2\cos\sigma\cos(\omega+\sigma))\omega_y$$

$$+ (\rho V(zt^{-1} + V\sin\omega) - N^2\cos\sigma\sin(\omega+\sigma))\omega_z - H(N_x\sin\sigma + N\cos\sigma\sigma_x) = 0; \tag{3.5}$$

$$HN\sin\sigma\,\omega_x + N^2\sin\sigma\cos(\omega+\sigma)\,\omega_y + N^2\sin\sigma\sin(\omega+\sigma)\,\omega_z$$

$$+ \rho\tilde{D}V + HN\sin\sigma\,\sigma_x - HN_x\cos\sigma + t^{-1}V\rho = 0. \tag{3.6}$$

Having made similar combinations of the remaining two projections of the induction equation, we have the following equations:

$$N\omega_t + (NU - HV\cos\sigma)\,\omega_x + N(V\sin\sigma\sin(\omega+\sigma) + yt^{-1})\,\omega_y$$

$$- N(V\sin\sigma\cos(\omega+\sigma) - zt^{-1})\,\omega_z + N\tilde{D}\sigma + HV_x\sin\sigma = 0; \tag{3.7}$$

$$HV\sin\sigma\,\omega_x + NV\cos\sigma\sin(\omega+\sigma)\,\omega_y$$

$$- NV\cos\sigma\cos(\omega+\sigma)\,\omega_z - \tilde{D}N + HV_x\cos\sigma - N(U_x + t^{-1}) = 0. \tag{3.8}$$

Finally, the last equation in (1.1) implies that

$$N(\sin(\omega+\sigma)\omega_y - \cos(\omega+\sigma)\,\omega_z) - H_x = 0. \tag{3.9}$$

The overdetermined system (3.3), (3.5)–(3.9) contains six equations for one function ω. We shall search for solutions of this system in which ω is determined with functional arbitrariness. For this, in the equations considered, all rank minors of the matrix composed of the coefficients of the derivatives of the function ω are set equal to zero. As in the cases of the Ovsyannikov plane and spherical vortices [12, 13], this condition is satisfied and leads to a nontrivial solution only for $\sigma = 0$ or $\sigma = \pi$. We assume that $\sigma = 0$ and N can have an arbitrary sign. In this case, the system of equations of the submodel reduces to the following. The invariant part of the system consists of Eqs. (3.2) and (3.4). For $\sigma = 0$, Eq. (3.6) reduces to the equation

$$\rho\tilde{D}V - HN_x + t^{-1}V\rho = 0.$$

In view of (3.3), from Eq. (3.8) we have

$$\tilde{D}N + NU_x - HV_x + N(hV + t^{-1}) = 0.$$

Finally, in view of (3.3), Eq. (3.9) is written as

$$H_x + hN = 0.$$

The noninvariant function ω is determined from Eqs. (3.3), (3.5), and (3.7). Eliminating the derivative ω_t, we obtain the classifying relation

$$(\rho V^2 - N^2)(H\omega_x + N(\cos\omega\,\omega_y + \sin\omega\,\omega_z)) = 0.$$

Below, only the case of zero second multiplier in this relation (the condition of conservation of ω along the magnetic lines) is considered. Let us derive the compatibility conditions for the equations for ω. Using the standard procedure, we obtain the following equations for the invariant functions:

$$N\tilde{D}h - HVh_x + t^{-1}hN = 0, \qquad Hh_x + h^2N = 0.$$

As in the Ovsyannikov plane vortex, for $h \neq 0$ we have the integral

$$H = t^{-1}H_0h$$

with an arbitrary constant H_0. Finally, the system of equations for the invariant functions is written as

$$\begin{aligned}
&\tilde{D}\rho + \rho(U_x + 2t^{-1} + hV) = 0,\\
&\tilde{D}U + \rho^{-1}p_x + \rho^{-1}NN_x = 0,\\
&\tilde{D}V - \rho^{-1}t^{-1}H_0hN_x + t^{-1}V = 0,\\
&\tilde{D}p + A(p,\rho)(U_x + 2t^{-1} + hV) = 0,\\
&\tilde{D}N - t^{-1}H_0hV_x + N(U_x + t^{-1} + hV) = 0,\\
&\tilde{D}h + Vh^2 + t^{-1}h = 0, \qquad t^{-1}H_0h_x + hN = 0.
\end{aligned} \tag{3.10}$$

The compatibility condition for the last two equations for the function h is the equation for N of the same system. Thus, system (3.10) is in involution.

Let us consider the system for the noninvariant function ω. As in the case of the Ovsyannikov plane and spherical vortices, the function ω is conserved along the particle trajectories and along the magnetic lines. For $h \neq 0$, the general solution of the subsystem for the noninvariant function has the form

$$F\Big(\frac{y}{t} - \frac{\cos\omega}{th}, \frac{z}{t} - \frac{\sin\omega}{t\,h}\Big) = 0 \tag{3.11}$$

(F is an arbitrary smooth function). Thus, the solution reduces to investigation of the system of equations in involution (3.10) and the final relation (3.11), which can be performed similarly to the investigation of the Ovsyannikov plane vortex [14].

3.2. *Submodel* $\{X_3+X_5, X_2-X_6, X_7\}$. The solution is represented as

$$u=U(t,x),\quad v=\frac{ty+z}{t^2+1}+V(t,x)\cos\omega(t,x,y,z),\quad w=\frac{tz-y}{t^2+1}+V(t,x)\sin\omega(t,x,y,z),$$

$$H=H(t,x),\quad K=N\cos\left(\omega(t,x,y,z)+\sigma(t,x)\right),\quad L=N\sin\left(\omega(t,x,y,z)+\sigma(t,x)\right),\tag{3.12}$$

$$p=p(t,x),\qquad \rho=\rho(t,x).$$

We substitute representation (3.12) into system (1.1). As above, by virtue of the continuity equation, the function $h(t,x)$ is defined as

$$\tilde{D}\rho+\rho\Big(U_x+\frac{2t}{t^2+1}+hV\Big)=0,\qquad \tilde{D}=\partial_t+U\,\partial_x.\tag{3.13}$$

For the function ω, we have

$$\sin\omega\,\omega_y-\cos\omega\,\omega_z+h=0.\tag{3.14}$$

The equations for the invariant functions are written as

$$\tilde{D}U+\rho^{-1}p_x+\rho^{-1}NN_x=0,$$

$$\tilde{D}H+H(2t(t^2+1)^{-1}+hV)=0,\tag{3.15}$$

$$\tilde{D}p+A(p,\rho)(U_x+2t(t^2+1)^{-1}+hV)=0.$$

From the remaining equations of system (1.1), we obtain an overdetermined system for the function ω. The momentum equation leads to

$$\rho V\omega_t+(\rho UV-HN\cos\sigma)\omega_x$$

$$+\left(\rho V((ty+z)(t^2+1)^{-1}+V\cos\omega)-N^2\cos\sigma\cos\left(\omega+\sigma\right)\right)\omega_y$$

$$+\left(\rho V((tz-y)(t^2+1)^{-1}+V\sin\omega)-N^2\cos\sigma\sin\left(\omega+\sigma\right)\right)\omega_z$$

$$-H(N_x\sin\sigma+N\cos\sigma\sigma_x)-\rho V(t^2+1)^{-1}=0;\tag{3.16}$$

$$HN\sin\sigma\,\omega_x+N^2\sin\sigma\cos\left(\omega+\sigma\right)\omega_y+N^2\sin\sigma\sin\left(\omega+\sigma\right)\omega_z$$

$$+\rho\tilde{D}V+HN\sin\sigma\,\sigma_x-HN_x\cos\sigma+t(t^2+1)^{-1}V\rho=0.\tag{3.17}$$

The induction equations imply the following relations:

$$N\omega_t+(NU-HV\cos\sigma)\,\omega_x+N((ty+z)(t^2+1)^{-1}+V\sin\sigma\sin\left(\omega+\sigma\right))\,\omega_y$$

$$-N(-(tz-y)(t^2+1)^{-1}+V\sin\sigma\cos\left(\omega+\sigma\right))\omega_z+N(\tilde{D}\sigma+(t^2+1)^{-1})+HV_x\sin\sigma=0;\tag{3.18}$$

$$HV\sin\sigma\,\omega_x+NV\cos\sigma\sin\left(\omega+\sigma\right)\omega_y$$

$$-NV\cos\sigma\cos\left(\omega+\sigma\right)\omega_z-\tilde{D}N+HV_x\cos\sigma-N(U_x+t(t^2+1)^{-1})=0.\tag{3.19}$$

The last equation of system (1.1) reduces to the form

$$N(\sin\left(\omega+\sigma\right)\omega_y-\cos\left(\omega+\sigma\right)\omega_z)-H_x=0.\tag{3.20}$$

As above, the function ω is determined with functional arbitrariness only for $\sin\sigma=0$. In this case, the equations reduce to the following. The invariant part of the submodel consists of Eqs. (3.13) and (3.15). For $\sin\sigma=0$, from Eq. (3.17), we obtain

$$\tilde{D}V-\rho^{-1}HN_x+t(t^2+1)^{-1}V=0.$$

From Eq. (3.19), we have

$$\tilde{D}N - HV_x + N(U_x + hV + t(t^2+1)^2) = 0.$$

Finally, from (3.20) we obtain

$$H_x + hN = 0.$$

The equations for the noninvariant function ω are derived from (3.14), (3.16), and (3.18). For $\sin\sigma = 0$, the last two equations have the form

$$\rho V\omega_t + (\rho UV - HN)\omega_x + (\rho V((ty+z)(t^2+1)^{-1} + V\cos\omega) - N^2\cos\omega)\omega_y$$

$$+ (\rho V((tz-y)(t^2+1)^{-1} + V\sin\omega) - N^2\sin\omega)\omega_z - (t^2+1)^{-1}\rho V = 0; \qquad (3.21)$$

$$N\omega_t + (NU - HV)\omega_x + N(ty+z)(t^2+1)^{-1}\omega_y + N(tz-y)(t^2+1)^{-1}\omega_z + (t^2+1)^{-1}N = 0. \qquad (3.22)$$

The compatibility condition for Eqs. (3.14) and (3.22) for the function ω are written in terms of only invariant functions:

$$N\tilde{D}h - HVh_x + t(t^2+1)^{-1}hN = 0.$$

From the compatibility condition for Eqs. (3.14) and (3.21), we obtain one more equation for the function ω:

$$((\rho V^2 - N^2)h\sin\omega + \rho V(t^2+1)^{-1}(t\sin\omega - 2\cos\omega))\omega_y$$

$$- ((\rho V^2 - N^2)h\cos\omega + \rho V(t^2+1)^{-1}(t\cos\omega + 2\cos\omega))\omega_z$$

$$- \rho V\tilde{D}h - HNh_x = 0. \qquad (3.23)$$

The last equation contains derivatives of ω only with respect to y and z. For $V \neq 0$, Eqs. (3.14) and (3.23) can be solved for these derivatives. Cross-differentiation of these equations leads to the condition of their compatibility in the form

$$((\rho V\tilde{D}h - HNh_x)\sin\omega + (\rho V^2 - N^2)h^2\sin\omega + \rho Vh(t^2+1)^{-1}(t\sin\omega - 2\cos\omega))^2$$

$$+ ((\rho V\tilde{D}h - HNh_x)\cos\omega + (\rho V^2 - N^2)h^2\cos\omega + \rho Vh(t^2+1)^{-1}(t\cos\omega + 2\cos\omega))^2 = 0. \qquad (3.24)$$

Equation (3.24) holds only for $h = 0$. Thus, from equations (3.14) and (3.23), it follows that $\omega_y = \omega_z = 0$, which implies that the examined partially invariant solution considered reduces to the invariant solution for the group $\{X_3 + X_5, X_2 - X_6\}$. A similar reduction is proved in [28] for the equations of gas dynamics in the absence of a magnetic field. Since the set of invariant solutions of the equations of magnetohydrodynamics (1.1) has generally been studied [18, 29], this solution is not investigated further.

3.3. *Submodel* $\{X_3, X_5, X_2 + X_6\}$. The solution is represented as

$$u = U(t,x), \quad w = y - tv + W(t,x), \quad \boldsymbol{B} = \boldsymbol{B}(t,x), \quad p = p(t,x), \quad \rho = \rho(t,x). \qquad (3.25)$$

Here the function $v = v(t,z,y,z)$ is noninvariant. The last equation in (1.1) implies that $H = H(t)$. Representation (3.25) is substituted into the continuity equation and split into invariant and noninvariant parts. For convenience, we introduce a new invariant function $h(t,x)$:

$$\tilde{D}\rho + \rho(U_x + h) = 0, \qquad \tilde{D} = \partial_t + U\,\partial_x.$$

For the noninvariant function v, we obtain the equation

$$v_y - tv_z = h. \qquad (3.26)$$

One more relation for the function v is found from the momentum equation in the projection onto the Oy axis:

$$\tilde{D}v + vv_y + (y + W - tv)v_z - \rho^{-1}HK_x = 0. \qquad (3.27)$$

The compatibility condition for Eqs. (3.26) and (3.27) is given by the following relation for v, which is linearly independent with these equations:

$$hv_y + (2 - th)v_z + \tilde{D}h = 0. \qquad (3.28)$$

This equation supplements the system of equations for the noninvariant function v. From Eqs. (3.26) and (3.28), it follows that the function v depends linearly on y and z, i.e., the solution defines the motion of a continuous medium with homogeneous deformation in the Oy and Oz directions (the strain rate tensor does not depend on the coordinates y and z).

The obtained class of solutions is described as follows:

$$u = U(t,x), \quad \bar{\boldsymbol{v}} = M(t,x)\bar{\boldsymbol{y}} + \bar{\boldsymbol{b}}(t,x), \quad \boldsymbol{B} = \boldsymbol{B}(t,x), \quad p = p(t,x), \quad \rho = \rho(t,x). \tag{3.29}$$

Here $\bar{\boldsymbol{v}} = (v,w)^{\mathrm{t}}$, $\bar{\boldsymbol{y}} = (y,z)^{\mathrm{t}}$, $\bar{\boldsymbol{b}} = (b^2,b^3)^{\mathrm{t}}$, and $\bar{\boldsymbol{B}} = (K,L)^{\mathrm{t}}$ are truncated vectors, and M is a square matrix of size 2×2. Substitution of the solution representation (3.29) into system (1.1) yields the following equations. The continuity equation and the pressure equation lead to

$$\tilde{D}\rho + \rho(U_x + \operatorname{tr} M) = 0, \qquad \tilde{D}p + A(p,\rho)(U_x + \operatorname{tr} M) = 0. \tag{3.30}$$

Projecting the momentum equation and the induction equation [the first and fourth equations of system (1.1)] onto the Ox axis, we have

$$\tilde{D}U + \frac{1}{\rho}\,\frac{\partial}{\partial x}\Big(p + \frac{1}{2}\,\bar{\boldsymbol{B}}^2\Big) = 0, \qquad \tilde{D}H + H(U_x + \operatorname{tr} M) - HU_x = 0. \tag{3.31}$$

Projecting the momentum equation onto the Oy and Oz axes, we obtain

$$(\tilde{D}M + M^2)\bar{\boldsymbol{y}} + M\,\bar{\boldsymbol{b}} + \tilde{D}\bar{\boldsymbol{b}} - \rho^{-1}H\bar{\boldsymbol{B}}_x = 0. \tag{3.32}$$

Similarly, projecting the induction equations onto the Oy and Oz axes, we have

$$\tilde{D}\bar{\boldsymbol{B}} + (U_x + \operatorname{tr} M)\bar{\boldsymbol{B}} - HM_x\bar{\boldsymbol{y}} - H\bar{\boldsymbol{b}}_x - M\bar{\boldsymbol{B}} = 0. \tag{3.33}$$

Finally, the last equation in (1.1) is simplified to the relation $H_x = 0$. Splitting of Eq. (3.32) with respect to $\bar{\boldsymbol{y}}$ leads to the relations

$$\tilde{D}M + M^2 = 0, \qquad \tilde{D}\bar{\boldsymbol{b}} + M\,\bar{\boldsymbol{b}} - \rho^{-1}H\bar{\boldsymbol{B}}_x = 0. \tag{3.34}$$

The general solution of the first equation in (3.34) is written as

$$M = (E + tM_0)^{-1}M_0, \qquad \tilde{D}M_0 = 0, \tag{3.35}$$

where E is a unit matrix. Thus,

$$\operatorname{tr} M = \tilde{D}\ln(j_2t^2 + j_1t + 1), \qquad j_1 = \operatorname{tr} M_0, \quad j_2 = \det M_0. \tag{3.36}$$

In Eq. (3.33), setting the coefficient at $\bar{\boldsymbol{y}}$ equal to zero leads to the dilemma: $H(t) \equiv 0$ or $M = M(t)$. Let us consider the first variant. In this case, integration of the second equation in (3.34) yields

$$\bar{\boldsymbol{b}} = (E + tM_0)^{-1}\bar{\boldsymbol{b}}_0, \qquad \tilde{D}\bar{\boldsymbol{b}}_0 = 0.$$

Using (3.30), from Eq. (3.33) we obtain $\bar{\boldsymbol{B}}$:

$$\bar{\boldsymbol{B}} = (\rho/\rho_0)(E + tM_0)^{-1}\bar{\boldsymbol{B}}_0, \qquad \tilde{D}\bar{\boldsymbol{B}}_0 = 0, \qquad \tilde{D}\rho_0 = 0.$$

Thus, in the case considered ($H = 0$), we obtain reduced equations of magnetohydrodynamics (1.1) of the form (3.29):

$$\begin{gathered}\tilde{D}\rho + \rho(U_x + \operatorname{tr} M) = 0, \qquad \tilde{D}p + A(p,\rho)(U_x + \operatorname{tr} M) = 0,\\ \tilde{D}U + \rho^{-1}(p + \bar{\boldsymbol{B}}^2/2)_x = 0, \qquad \tilde{D}\psi = 0,\\ M = (E + tM_0)^{-1}M_0, \qquad \bar{\boldsymbol{B}} = \rho(E + tM_0)^{-1}\bar{\boldsymbol{B}}_0/\rho_0.\end{gathered} \tag{3.37}$$

Here M_0, $\bar{\boldsymbol{B}}_0$, and ρ_0 are a matrix, vector, and scalar, respectively, which depend arbitrarily on ψ.

Let us consider the second variant [$M = M(t)$]. In this case, in representation (3.35) M_0 is an arbitrary constant matrix. Using expressions (3.36), we integrate the second equation in (3.31):

$$H = H_0/(j_2t^2 + j_1t + 1).$$

In the case considered, the equations of the submodel become

$$\tilde{D}\rho + \rho(U_x + \operatorname{tr} M) = 0, \qquad \tilde{D}p + A(p,\rho)(U_x + \operatorname{tr} M) = 0,$$
$$\tilde{D}U + \rho^{-1}(p + \bar{\boldsymbol{B}}^2/2)_x = 0, \qquad \tilde{D}\bar{\boldsymbol{B}} + (U_x + \operatorname{tr} M)\bar{\boldsymbol{B}} - H\bar{\boldsymbol{b}}_x - M\bar{\boldsymbol{B}} = 0, \tag{3.38}$$
$$\tilde{D}\bar{\boldsymbol{b}} + M\bar{\boldsymbol{b}} - \rho^{-1}H\bar{\boldsymbol{B}}_x = 0.$$

The solution with homogeneous deformation in y and z of the form (3.29) is described by system (3.37), (3.38) for the cases $H = 0$ and $H \neq 0$, respectively. We note that, in both cases, the solution contains a singularity at the time $t = -1/\lambda$ (λ is the eigenvalue of the matrix M_0).

Conclusions. The above analysis completes the investigation of the compatibility conditions of partially invariant submodels of defect 1 of the equations of ideal magnetohydrodynamics. The further analysis of these systems of equations can involve calculations of the symmetry group admitted by the constructed submodels and finding classes of their exact solutions. In this case, most of the admitted group is inherited from the initial model, but extensions of the initial group compared to the inherited one are possible. In the present work, regular partially invariant submodels of defects 2 and 3 and the class of barochronic solutions for the equations of ideal magnetohydrodynamics were not investigated.

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